Numerical Solution of Stiff Delay and Singular Delay Systems using Leapfrog Method

S. Sekar and M. Vijayarakavan

Abstract— In this paper, a new method of analysis of the stiff delay and singular delay systems using Leapfrog method is presented. To illustrate the effectiveness of the Leapfrog method, an example of the stiff delay and singular delay systems has been considered and the solutions were obtained using methods taken from the literature (single-term Haar wavelet series [6]) and Leapfrog method and are compared with the exact solutions of the stiff delay and singular delay systems. Error graphs are presented to highlight the efficiency of the Leapfrog method.

Index Terms— Haar wavelet, Single term Haar wavelet series, Leapfrog method, singular delay systems, stiff delay systems.

1 Introduction

A system of DDEs is considered stiff when it contains processes of widely different time scales. From a computational point of view the stiffness implies that, while solving numerically the corresponding initial value problem by a given method with assigned tolerance, a stepsize is restricted by stability requirements rather than by the accuracy demands. The response of an immune system cannot be represented correctly without the hereditary phenomena being taken into account: cell division, differentiation, etc. The kinetic parameters of the models represent high-rate (molecular) and slowrate (cellular) interactions in the immune system that span a time scale from seconds to days. Therefore, the systems of DDEs appearing in immune response modelling are typically stiff.

Now day, the phenomena of time delay in many practical systems have had many scholars' much attention. And many excellent results for the systems with time delay have been obtained [1–18]. Especially for the relationship between eigenvalue and stability of differential systems with delay, much achievement has been gotten. But we notice that a lot of practical systems, such as economic systems, power systems and so on, are singular differential systems with delay. In [10–18], authors have discussed the singular differential systems, even the singular differential systems with delay, and have gotten some consequences. But up to now, for the relationship between eigenvalue and stability of singular differential systems with delay, there is hardly effective verdict.

In this article introduced a numerical method for addressing stiff delay and singular delay systems by an application of the Leapfrog method which was studied by S.Sekar and team of his researchers [4-5, 7-9]. In Section 2, the Leapfrog method for solving stiff delay and singular delay systems is introduced. In Section 3 presents general form of stiff delay and singular delay systems. In Section 4 and 5, the Leapfrog and STHWS [4-5, 7-9] method for solving stiff delay and singular delay systems is solved. We refer [4-9] for the numerical treatment of stiff delay and singular delay systems.

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2 Leapfrog Method

In mathematics Leapfrog integration is a simple method for numerically integrating differential equations of the form $\ddot{x} = F(x)$, or equivalently of the form $\dot{v} = F(x), \dot{x} \equiv v$, particularly in the case of a dynamical system of classical mechanics. Such problems often take the form $\ddot{x} = -\nabla V(x)$, with energy function $E(x,v) = \frac{1}{2}|v|^2 + V(x)$, where V is the potential energy of the system. The method is known by different names in different disciplines. In particular, it is similar to the Velocity Verlet method, which is a variant of Verlet integration. Leapfrog integration is equivalent to updating positions $x(t)$ and velocities $v(t) = \dot{x}(t)$ at interleaved time points, staggered in such a way that they 'Leapfrog' over each other. For example, the position is updated at integer time steps and the velocity is updated at integer-plus-a-half time steps.

Leapfrog integration is a second order method, in contrast to Euler integration, which is only first order, yet requires the same number of function evaluations per step. Unlike Euler integration, it is stable for oscillatory motion, as long as the time-step Δt is constant, and $\Delta t \le 2/w$. In Leapfrog integration, the equations for updating position and velocity are

$$x_i = x_{i-1} + v_{i-1/2}\Delta t,$$
$$a_i = F(x_i)$$
$$v_{i+1/2} = v_{i-1/2} + a_i \Delta t,$$

where x_i is position at step $i, v_{i+1/2}$, is the velocity, or first derivative of x, at step $i+1/2, a_i = F(x_i)$ is the acceleration, or second derivative of x, at step i and Δt is the size of each time step. These equations can be expressed in a form which gives

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velocity at integer steps as well. However, even in this synchronized form, the time-step Δt must be constant to maintain stability.

$$x_{i+1} = x_i + v_i \Delta t + \frac{1}{2} a_i \Delta t^2,$$

$$v_{i+1} = v_i + \frac{1}{2}(a_i + a_{i+1})\Delta t.$$

One use of this equation is in gravity simulations, since in that case the acceleration depends only on the positions of the gravitating masses, although higher order integrators (such as Runge–Kutta methods) are more frequently used. There are two primary strengths to Leapfrog integration when applied to mechanics problems. The first is the time-reversibility of the Leapfrog method. One can integrate forward *n* steps, and then reverse the direction of integration and integrate backwards *n* steps to arrive at the same starting position. The second strength of Leapfrog integration is its symplectic nature, which implies that it conserves the (slightly modified) energy of dynamical systems. This is especially useful when computing orbital dynamics, as other integration schemes, such as the Runge-Kutta method, do not conserve energy and allow the system to drift substantially over time.

3 General Form of Stiff Delay and Singular Delay Systems

In this section, the stiff delay system of the form is considered

$$\dot{x}(t) = Ax(t) + Bx(t-h)$$
$$x(t) = \phi(t)$$
$$t \in [-h, 0]$$

where x is an n-state vector and A and B are $n \times n$ matrices. $\phi(t)$ is an initial function.

And the singular delay system of the form is considered

$$K\dot{x}(t) = Ax(t) + Bx(t-h) + Cu(t)$$
$$x(t) = \phi(t) \text{ on } [-h,0]$$

where K is an $n \times n$ singular matrix, A and B are $n \times n$ matrices and C is an $n \times m$ matrix. $\phi(t)$ is an initial function, x is an n-state vector and u is an m-vector of control.

To highlight the efficiency of the Leapfrog method, we consider the following two different examples taken from the real world applications (4 Example and 5 Example), along with the exact solutions. The discrete solutions obtained by the two methods, Leapfrog method and STHWS method, the absolute errors between them are calculated. To distinguish the effect of the errors in accordance with the exact solutions, a graphical representation is given for selected values of " x "and is presented in Figures 1 - 4 and Table 1 - 4 for stiff delay and singular delay systems, using three-dimensional effect.

4 Example of Stiff Delay Systems

Consider the stiff delay system S. Sekar et al. [6]

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 11 & 21 \\ -21/2 & -41/2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} q-2p & 2p-2q \\ p-q & p-2q \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix}$$

for $0 < t \le 4$ and $p = e^{-\frac{1}{2}}$ and $q = e^{-11}$ with $x(t) = [1 \quad 0]^T$ on $[-1 \quad 0]$. The exact solution of (4) is

$$\left.\begin{aligned} x_1(t) &= 2\exp\left(-\frac{t}{2}\right) - \exp(-11t) \\ x_2(t) &= -\exp\left(-\frac{t}{2}\right) + \exp(-11t) \end{aligned}\right\}$$

the discrete time solution and the exact solution are evaluated. The results are shown in Table 1-2 and Fig. 1-2.

TABLE 1
Error Calculation for $x_1(t)$

t	Example 4				
	Exact	STHWS		Leapfrog	
	Solution	Solution	Error	Solution	Error
0.1	1.569587765	1.569589365	1.6E-06	1.569587781	1.6E-08
0.2	1.698871678	1.698874878	3.2E-06	1.69887171	3.2E-08
0.3	1.684532785	1.684535685	2.9E-06	1.684532814	2.9E-08
0.4	1.625184166	1.625186366	2.2E-06	1.625184188	2.2E-08
0.5	1.553514795	1.553516195	1.4E-06	1.553514809	1.4E-08
0.6	1.480276073	1.480276973	9E-07	1.480276082	9E-09
0.7	1.408923352	1.408923552	2E-07	1.408923354	2E-09
0.8	1.340489359	1.340585359	9.6E-05	1.340490319	9.6E-07
0.9	1.275206129	1.275295129	8.9E-05	1.275207019	8.9E-07
1.0	1.213044618	1.213127618	8.3E-05	1.213045448	8.3E-07

TABLE 2
Error Calculation for $x_2(t)$

t	Example 4				
	Exact	STHWS		Leapfrog	
	Solution	Solution	Error	Solution	Error
0.1	-0.618358341	-0.618356041	2.3E-06	-0.618358318	2.3E-08
0.2	-0.79403426	-0.79403156	2.7E-06	-0.794034233	2.7E-08
0.3	-0.823824809	-0.823821709	3.1E-06	-0.823824778	3.1E-08
0.4	-0.806453413	-0.806449913	3.5E-06	-0.806453378	3.5E-08
0.5	-0.774714012	-0.774710112	3.9E-06	-0.774713973	3.9E-08
0.6	-0.739457853	-0.739453553	4.3E-06	-0.73945781	4.3E-08
0.7	-0.704235263	-0.704230563	4.7E-06	-0.704235216	4.7E-08
0.8	-0.670169313	-0.670164213	5.1E-06	-0.670169262	5.1E-08
0.9	-0.637577977	-0.637572477	5.5E-06	-0.637577922	5.5E-08
1.0	-0.606513958	-0.606508058	5.9E-06	-0.606513899	5.9E-08

5 Example of Singular Delay Systems

Consider the singular delay system S. Sekar et al. [6]

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -\pi/2e & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with $x(t) = [1 \quad e-1]^T$ on $[-1 \quad 0]$. The exact solution of (6) is

$$x_1(t) = \exp(-t)\sin(\pi t/2)$$
$$x_2(t) = \exp(-t)\sin(\pi t/2) + \exp(-t+1)\cos(\pi t/2) - 1$$

the discrete time solution and the exact solution are evaluated. The results are shown in Table 3-4 and Fig. 3-4.

FIG. 1. ERROR ESTIMATION OF EXAMPLE 4 AT $x_1(t)$

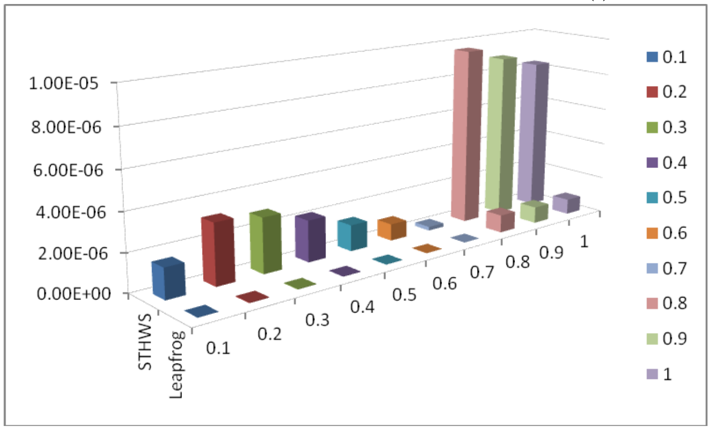


TABLE 3
ERROR CALCULATION FOR $x_1(t)$

t	Example 5				
	Exact	STHWS		Leapfrog	
	Solution	Solution	Error	Solution	Error
0.1	0.372900226	0.372904826	4.6E-06	0.372900272	4.6E-08
0.2	-0.614856354	-0.614851454	4.9E-06	-0.614856305	4.9E-08
0.3	0.708500714	0.708505914	5.2E-06	0.708500766	5.2E-08
0.4	-0.664809247	-0.664803747	5.5E-06	-0.664809192	5.5E-08
0.5	0.516099076	0.516104876	5.8E-06	0.516099134	5.8E-08
0.6	-0.306669932	-0.306663832	6.1E-06	-0.306669871	6.1E-08
0.7	0.083106381	0.083112781	6.4E-06	0.083106445	6.4E-08
0.8	0.114050189	0.114056889	6.7E-06	0.114050256	6.7E-08
0.9	-0.256093348	-0.256086348	7E-06	-0.256093278	7E-08
1.0	0.328882993	0.328890293	7.3E-06	0.328883066	7.3E-08

TABLE 4
ERROR CALCULATION FOR $x_2(t)$

t	Example 5				
	Exact	STHWS		Leapfrog	
	Solution	Solution	Error	Solution	Error
0.1	-2.868118601	-2.868118001	6E-07	-2.868118595	6E-09
0.2	-0.145294494	-0.145293394	1.1E-06	-0.145294483	1.1E-08
0.3	-0.879794603	-0.879793003	1.6E-06	-0.879794587	1.6E-08
0.4	-1.897974291	-1.897972191	2.1E-06	-1.89797427	2.1E-08
0.5	0.382208613	0.382211213	2.6E-06	0.382208639	2.6E-08
0.6	-2.543854823	-2.543851723	3.1E-06	-2.543854792	3.1E-08
0.7	0.413927565	0.413931165	3.6E-06	0.413927601	3.6E-08
0.8	-2.067352348	-2.067348248	4.1E-06	-2.067352307	4.1E-08
0.9	-0.397722588	-0.397717988	4.6E-06	-0.397722542	4.6E-08
1.0	-1.119190623	-1.119185523	5.1E-06	-1.119190572	5.1E-08

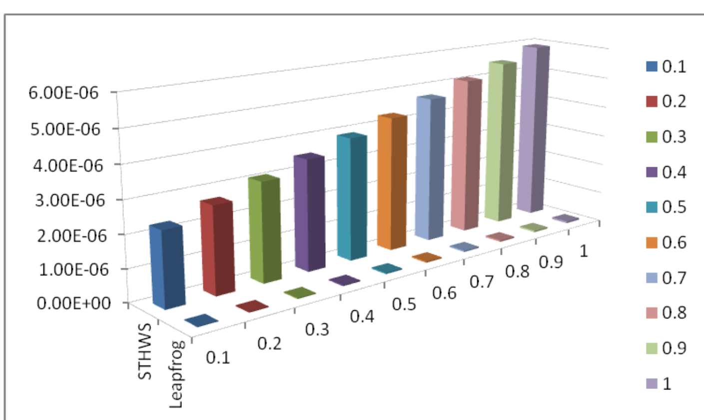


FIG. 2. ERROR ESTIMATION OF EXAMPLE 4 AT $x_2(t)$

FIG. 3. ERROR ESTIMATION OF EXAMPLE 5 AT $x_1(t)$

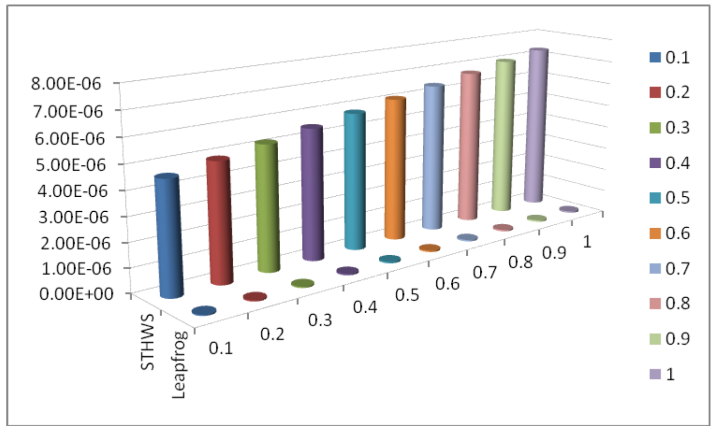
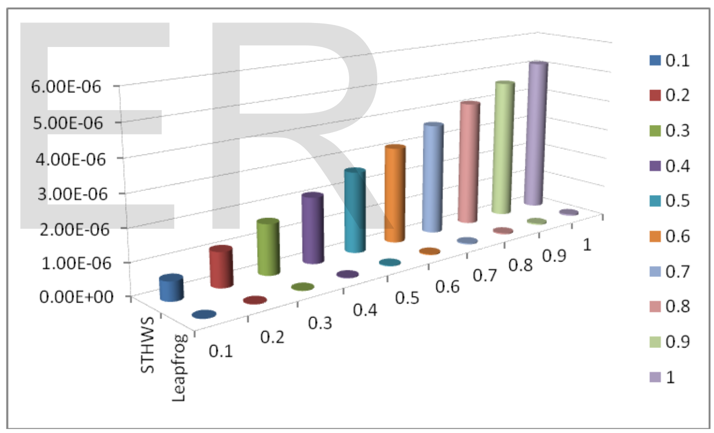


FIG. 4. ERROR ESTIMATION OF EXAMPLE 5 AT $x_2(t)$



6 CONCLUSION

The obtained discrete solution of the numerical examples shows the efficiency of the Leapfrog method for solving the stiff delay and singular delay systems. From the Figures 1 - 4, we can observe that for most of the problems, the absolute error is less (almost no error) in Leapfrog method when compared to the STHWS [6] which yields a little error, along with the exact solutions. From the Figures 1 - 4 and Tables 1 - 4, one can predict that the error is very less in Leapfrog method when compared to STHWS method [6]. Hence, the Leapfrog method is more suitable for studying the stiff delay and singular delay systems.

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